MA114 Summer 2018
Worksheet 14 – Power Series Part 2 – 7/05/18

1. Using the fact that $\dfrac{-2}{3x^2+4x+1} = \dfrac{1}{1+x} - \dfrac{3}{1+3x}$, find a power series expression for $\dfrac{1}{3x^2+4x+1}$ around $x = 0$.

$$\frac{1}{3x^2+4x+1} = -\frac{1}{2}\left(\frac{-2}{3x^2+4x+1}\right) = -\frac{1}{2}\left(\frac{1}{1+x} - \frac{3}{1+3x}\right) = -\frac{1}{2}\left(\frac{1}{1-(-x)} - 3\cdot\frac{1}{1-(-3x)}\right)$$

$$= -\frac{1}{2}\left(\sum_{n=0}^{\infty}(-x)^n - 3\sum_{n=0}^{\infty}(-3x)^n\right)$$

$$= -\frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}(-3)^{n+1}x^n\right)$$

$$= \boxed{\sum_{n=0}^{\infty} -\frac{1}{2}\left((-1)^n + (-3)^{n+1}\right)x^n}$$

2. Use the same idea as above to give a series expression for $\ln(1+2x)$ given that $\int \dfrac{2\,dx}{1+2x} = \ln(1+2x)$. You will want to manipulate the fraction $\dfrac{2}{1+2x} = \dfrac{2}{1-(-2x)}$ as above.

$$\ln(1+2x) = \int \frac{2\,dx}{1+2x} = 2\int \frac{1}{1-(-2x)}dx = 2\int\left(1 - 2x + (-2x)^2 + (-2x)^3 + \dots\right)dx$$

$$= \boxed{2\left(x - \frac{2x^2}{2} + \frac{4x^3}{3} + \frac{-8x^4}{4} + \dots\right)}$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n-1}x^n}{n}$$

$$= \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n x^n}{n}}$$

3. Write $(1+x^2)^-2$ as a power series. Hint: Use term-by-term differentiation.

$$\left(\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\right) = \frac{1}{(1+x^2)^2}\cdot -2x$$

$$\frac{1}{(1+x^2)^{-2}} = \frac{-1}{2x}\cdot\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$$= -\frac{1}{2x}\cdot\frac{d}{dx}\left(\frac{1}{1-(-x^2)}\right)$$

$$= -\frac{1}{2x}\cdot\sum_{n=0}^{\infty}(-x^2)^n$$

(My bad.)

So not quite a power series.

$$= \sum_{n=0}^{\infty} -\frac{1}{2x}\cdot(-1)^n x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2}x^{2n-1} = \boxed{\frac{-1}{2x} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2}x^{2n-1}}$$